Astro 501, Fundamental Astronomy — Problem Set 3

Due Sept 25

1. A star with an apparent magnitude of $m_{AB} = 20$ in the J-band is imaged with a 4-m telescope and a narrow-band (100 Å wide) filter centered at a wavelength of 1.2 microns. Due to seeing, the star's image extends over a diameter 1.5 arcseconds. The sky brightness in the J-band is $m_{AB} = 16.5$ mag arcsec^{-2}. Assume that the atmosphere + telescope + filter + detector throughtput is 0.3 (i.e., 30% of all photons are detected), that each photon detected produces one count in the detector, and there is no noise associated with the detector itself.

a) What is the count rate on the detector from the star? What is the count rate produced by the sky which is underlying the star?

Flux density is related to AB magnitude via

$$F_\nu = 3.63 \times 10^{-20} \cdot 10^{-m_{AB}/2.5} \text{ ergs cm}^{-2}\text{ s}^{-1}\text{ Hz}^{-1}$$

Since we're using a $\Delta\lambda = 100$ Å filter, it is best to convert F_ν to F_λ,

$$F_\lambda = F_\nu \frac{c}{\lambda^2}$$

The total flux in the bandpass is therefore

$$F_{\text{tot}} = F_\lambda \cdot \Delta\lambda$$

Finally, to go from ergs cm^{-2} s^{-1} Å^{-1} to photon flux (in photons per second), we simply divide by the energy per photon, multiply by the area of the telescope, and, since the total system throughput is 30%, multiply by $\epsilon = 0.3$. Thus

$$N_{\text{ph}} = 3.63 \times 10^{-20} \cdot 10^{-m_{AB}/2.5} \left(\frac{c}{\lambda^2}\right) \left(\frac{\lambda}{hc}\right) \Delta\lambda \cdot \epsilon \cdot \left(\frac{\pi D^2}{4}\right)$$

where $D = 400$ cm is the size of the telescope and $\lambda = 12000$ Å. Plugging in the numbers yields a count rate of $N = 17.2$ counts/sec from the star, and $n_s = 432.3$ counts/sec/arcsec^2 from the underlying sky. Since the star covers an area of $A = \pi(1.5/2)^2$ arcsec^2, this gives a total sky count rate of $n = n_s A = 764.0$ counts/sec.

b) Presumably, the mean brightness of the sky is well measured from other regions of the detector. This being the case, how long an exposure would it take to measure the star's brightness to a precision of 0.02 mag? (Note and quick exercise to the student: for small numbers, say, $\Delta m < 0.3$ mag, an uncertainty of 0.XX in magnitudes corresponds to an uncertainty of about XX% in flux.)

A precision of 0.02 mag implies a flux (or count) error of $\sim 2\%$. In other words, you want the measurement of the star (the signal) to be 50 times that of the noise. After integrating for t seconds, the number of counts in the star and the sky will be $C_{\text{star}} = Nt$ and $C_{\text{sky}} = nt$,

respectively. From photons statistics, the uncertainy in a measurement is the square root of the number of counts. Since the signal is Nt, the signal-to-noise will be

$$\text{SNR} = \frac{Nt}{\sqrt{Nt+nt}} \implies t = \frac{(\text{SNR})^2(N+n)}{N^2}$$

Plugging in the numbers for $N = 17.2$ and $n = 764.0$ yields an exposure time of 6593 seconds, or 1.83 hours.

c) Suppose that instead of being seeing limited, the observations were diffraction limited. Once again, how long would it take to measure the star to a precision of 0.02 mag? Is the use of adaptive optics to deliver diffraction-limited images worth it for this project?

The diffraction limit of a 4-m telescope at $1.2\ \mu$ is

$$\theta = 1.22\,\frac{\lambda}{D} = 3.66 \times 10^{-7}\ \text{radians} = 0.075\ \text{arcsec}$$

In this case, the number of counts in the sky underlying the star is

$$n = n_s \cdot \pi\theta^2 = 7.74\ \text{photons s}^{-1}$$

The time to achieve a signal-to-noise of 50 is then

$$\text{SNR} = \frac{Nt}{\sqrt{Nt+nt}} \implies t = \frac{(\text{SNR})^2(N+n)}{N^2}$$

or 210 seconds. With adaptive optics, the observations would be ~ 30 times faster for this project.

2. Write an equation for the semi-amplitude of a single-line spectroscopic binary (in km/s) as a function of the primary mass (with units of solar masses), the period (in days), and the mass ratio, q. (Such an equation is very useful for quick calculations.) What is the coefficient out front? What is the maximum K velocity which can be induced by an object at the star/planetary boundary ($\mathcal{M} = 0.08\,\mathcal{M}_\odot$) orbiting a solar-type star with a 1 year period? (Following the results from the Kepler satellite, you may figure than $\epsilon_{\max} \sim 0.9$.)

The semi-amplitude of a spectroscopic binary is

$$K_1 = \left(\frac{2\pi}{P}\right) \frac{a_1 \sin i}{\sqrt{1-\epsilon^2}}$$

Substituting

$$a_1 = a \left(\frac{\mathcal{M}_2}{\mathcal{M}_1 + \mathcal{M}_2}\right)$$

and

$$a^3 = k\,(\mathcal{M}_1 + \mathcal{M}_2)\,P^2$$

then yields

$$K_1 = \frac{2\pi}{P} \left\{k\,P^2\,(\mathcal{M}_1 + \mathcal{M}_2)\right\}^{1/3} \frac{\sin i}{\sqrt{1-\epsilon^2}} \left(\frac{\mathcal{M}_2}{\mathcal{M}_1 + \mathcal{M}_2}\right)$$

Collecting terms gives

$$K_1 = 2\pi k^{1/3} \frac{\sin i}{\sqrt{1-\epsilon^2}} P^{-1/3} \mathcal{M}_2\,(\mathcal{M}_1 + \mathcal{M}_2)^{-2/3}$$

Finally, substituting $q = \mathcal{M}_2/\mathcal{M}_1$ gives

$$K_1 = 2\pi k^{1/3} \frac{\sin i}{\sqrt{1-\epsilon^2}} \left(\frac{\mathcal{M}_1}{P}\right)^{1/3} \frac{q}{(1+q)^{2/3}}$$

Plugging in the numbers for units of solar masses and days gives

$$K_1 = 212.9 \left(\frac{\mathcal{M}}{P}\right)^{1/3} \frac{q}{(1+q)^{2/3}} \frac{\sin i}{\sqrt{1-\epsilon^2}} \text{ km s}^{-1}$$

A $\mathcal{M} = 0.08\mathcal{M}_\odot$ object in an eccentric orbit would induce a K velocity of about ~ 5.2 km s^{-1}.

3. A star is found to be a double-lined spectroscopic binary, with an eccentricity, $\epsilon = 0.5$. Spectroscopic measurements show the radial velocity curves of both components are near sinusoids, with $K_1 = 26.05$ km s^{-1}, $K_2 = 27.40$ km s^{-1}, and $P = 104.0233$ days.

a) Based on these data, what are the minimum masses on the components.

The ratio of the two masses is easily found through the ratio of the two K velocities:

$$\frac{\mathcal{M}_2}{\mathcal{M}_1} = \frac{K_1}{K_2}$$

Thus, the ratio of the two masses is $q = 0.95$. Also, from the previous problem, the K velocity of a system is given by

$$K_1 = 212.9 \left(\frac{\mathcal{M}_1}{P}\right)^{1/3} \frac{q}{(1+q)^{2/3}} \frac{\sin i}{\sqrt{1-\epsilon^2}} \text{ km s}^{-1}$$

So solving for $\mathcal{M}_1$ gives

$$\mathcal{M}_1 \sin^3 i = \frac{k}{(2\pi)^3} K_1^3 \, P \, \frac{(1+q)^2}{q^3} \left(1-\epsilon^2\right)^{3/2} = 0.548 \mathcal{M}_\odot$$

The mass of the secondary then comes from the ratio of the two K values

$$\mathcal{M}_2 = \left(\frac{K_1}{K_2}\right) \mathcal{M}_1 = 0.521 \mathcal{M}_\odot$$

b) A separate series of interferometric observations resolves the two components of the binary, and finds that the system's projected semi-major and semi-minor axes are $a = 56.47$ milli-arcsec, and $b = 41.42$ milli-arcsec. The Hipparchos parallax of the system is 77.3 milli-arcsec. What are the masses of the two stars? What are the inclinations?

For visual binaries, we have a relation that involves the cosine of the angle and the distance. Specifically,

$$\mathcal{M}_1 + \mathcal{M}_2 = \left(\frac{d\alpha}{\cos i}\right)^3 \frac{1}{kP^2} \implies \mathcal{M}_1(1+q) = \left(\frac{\alpha}{p \cos i}\right)^3 \frac{1}{kP^2}$$

where d is the distance to the system, α is the angular semi-major axis, and p is the parallax angle. Plugging in the numbers (while converting parsecs to Astronomical Units) gives

$$\mathcal{M}_1 \cos^3 i = \left(\frac{\alpha}{p}\right)^3 \frac{1}{kP^2} \frac{1}{1+q} = 4.806 \mathcal{M}_\odot$$

Combining with part (a) yields

$$\tan^3 i = \frac{\sin^3 i}{\cos^3 i} = \frac{0.548 \mathcal{M}_\odot}{4.806 M_\odot} = 0.114$$

So $i = 26^\circ$, $M_1 = 6.6 M_\odot$, and $M_2 = 6.3 M_\odot$.